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Integer-antimagic spectra of disjoint unions of cycles

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Integer-antimagic spectra of disjoint unions of cycles

Cover Page Footnote
I would like to thank my undergraduate student Mr. Samuel Lo helping me to do the preliminary work.

1 Introduction and some useful lemmas

Let G be a simple graph. For any nontrivial abelian group A (written additively), let $A^* = A\backslash\{0\}$, where 0 is the additive identity of A. Let a mapping $f : E(G) \to A^*$ be an edge labeling of G and $f^+ : V(G) \to A$ be its induced labeling, which is defined by $f^+(v) = \sum\limits_{uv\in E(G)} f(uv)$. If there exists an edge labeling f whose induced labeling f^+ on $V(G)$ is injective, then we say that f is an A-*antimagic labeling* and that G is an A-*antimagic* graph. The *integer-antimagic spectrum* of a graph G is the set $\mathrm{IAM}(G) = \{k \mid G \text{ is } \mathbb{Z}_k\text{-antimagic and } k \geq 2\}$. Clearly $\mathrm{IAM}(G) \subseteq \{k \mid k \geq |V(G)|\}$. In this paper, we will use the term 'labeling' instead of edge labeling. Notation and concepts not defined here are referred to the book [1].

The concept of the A-antimagicness property for a graph G (introduced in [2]) naturally arises as a variation of the A-magic labeling problem (where the induced vertex labeling is a constant map) (for example, see [4–6, 8–10, 13]). The concept of ring-magic is another variation of group-magic (see [7]). It is also a variation of the anti-magic labeling problem (for example, see [16]) and edge-graceful labeling problem (for example, see [17]). The integer-antimagic spectra of some famous classes of graphs were determined [2, 3, 11, 12, 14, 15].

The following result is obvious.

Lemma 1.1 ([2]). *Let $G = (V, E)$ be a simple graph and A be any abelian group. Let $f : E \to A^*$ Then*

$$\sum_{v\in V} f^+(v) = \sum_{v\in V}\sum_{vu\in E} f(vu) = 2\sum_{e\in E} f(e). \tag{1.1}$$

By using (1.1) and letting $A = \mathbb{Z}_{4m+2}$ for some $m \geq 0$, we have

Lemma 1.2 ([2, Lemma 1]). *For $m \geq 1$, a graph of order $4m + 2$ is not $\mathbb{Z}_{4m+2}$-antimagic.*

For integers $a \leq b$, let $[a, b]$ denote the set of integers from a to b, inclusive. We shall still call it an *interval*.

Proposition 1.3. *All elements in $[a, b]$ are distinct after taking modulo k for $k \geq b - a + 1$.*

Let $\mathbb{N}$ be the set of all positive integers. Let $f : E(G) \to \mathbb{N}$ be a labeling of a graph G and f^+ be its induced vertex labeling. Denote the range of f^+ as

$$I_f(G) = \{f^+(v) \mid v \in V(G)\}.$$

Suppose M is the maximum value (edge label) of f. Sometimes we will refer to the range of f^+ and the maximum edge label of f in the same concise notation by

$$I_f(G) = \{f^+(v) \mid v \in V(G)\} \triangleleft (M).$$

Lemma 1.4 ([14, Corollary 2.6]). *For $n \geq 1$, there is a labeling f for each of the following cycles such that $I_f(C_{4n-1}) = [3, 4n+1] \triangleleft (2n+1)$, $I_f(C_{4n}) = [3, 4n+2] \triangleleft (2n+1)$, $I_f(C_{4n+1}) = [2, 4n + 2] \triangleleft (2n + 1)$ and $I_f(C_{4n+2}) = [3, 4n + 5] \setminus \{4n + 2\} \triangleleft (2n + 3)$.*

Proof. As a self-contained paper, we list the labelings for cycles defined in [2].

Let $C_p = u_1u_2 \cdots u_nu_1$ and $e_1 = u_1u_2$, $e_2 = u_2u_3$, $\ldots$, $e_p = u_pu_1$ be its edges.

Case 1. $p = 4n, n \geq 1$:

$$f(e_i) = \begin{cases} i & \text{if } 1 \leq i \leq 2n; \\ 3 + 2(2n - \lceil \frac{i}{2} \rceil) & \text{if } 2n+1 \leq i \leq 4n. \end{cases}$$

Case 2. $p = 4n+1, n \geq 1$:

$$f(e_i) = \begin{cases} i & \text{if } 1 \leq i \leq 2n; \\ 3 + 2(2n - \lceil \frac{i}{2} \rceil) & \text{if } 2n+1 \leq i \leq 4n+1. \end{cases}$$

Case 3. $p = 4n+2, n \geq 1$:

$$f(e_i) = \begin{cases} i & \text{if } 1 \leq i \leq 2n+3; \\ 3 + 2(2n - \lceil \frac{i-2}{2} \rceil) & \text{if } 2n+4 \leq i \leq 4n+2. \end{cases}$$

Case 4. $p = 4n-1, n \geq 2$:

$$f(e_i) = \begin{cases} i & \text{if } 1 \leq i \leq 2n+1; \\ 3 + 2(2n - \lceil \frac{i+1}{2} \rceil) & \text{if } 2n+2 \leq i \leq 4n-1. \end{cases}$$

Case 5. $p = 3$: We label the edges of C_3 by $1, 2$ and 3. Hence $I_f(C_3) = [3, 5]$.

By using each labeling defined above, we have the lemma. $\square$

For $S \subset \mathbb{Z}$ and $a \in \mathbb{Z}$, we define the set $a + S = \{a + s \mid s \in S\}$.

Lemma 1.5 ([12, Lemma 3.1])**.** *For $n \geq 3$, suppose $g : E(C_n) \to \mathbb{Z}$ is a labeling and $c \in \mathbb{Z}$. Let $h = g + c$. Then $I_h(C_n) = 2c + I_g(C_n)$.*

Corollary 1.6. *Let G be a disjoint union of cycles. Suppose $g : E(G) \to \mathbb{Z}$ is a labeling and $c \in \mathbb{Z}$. Let $h = g + c$. Then $I_h(G) = 2c + I_g(G)$.*

Lemma 1.7. *For $n \geq 1$, there is a labeling h for C_{4n+2} such that $I_h(C_{4n+2}) = [2, 4n+4] \setminus \{4n+1\} \lhd (2n+3)$.*

Proof. Consider the labeling f for C_{4n+2} defined in Lemma 1.4. We reduce the label each edge e_{2j} by 1. Let the resulting labeling be h. Then $I_h(C_{4n+2}) = [2, 4n+4] \setminus \{4n+1\}$. Clearly the maximum edge label is still $2n+3$. $\square$

2 Disjoint unions of cycles of orders congruent to 0 modulo 4

Let G and H be two disjoint graphs. Let $G + H$ be the disjoint union of G and H. The disjoint union of n copies of a graph G is denoted by nG.

Suppose g and h are labelings of graphs G and H, respectively. The labeling $g \cup h$ of $G + H$ is the labeling obtained by combining g and h.

Applying Corollary 1.6 we have

Lemma 2.1. *Let G be an even order disjoint union of cycles and let H be another disjoint union of cycles. Suppose there are labelings g and h of G and H such that $I_g(G) = [3, |G|+2] \lhd (m)$ and $I_h(H) = [3, |H|+2] \lhd (M)$, where $m \leq |G|/2$. Let $f = g \cup (h + |G|/2)$. Then $I_f(G+H) = [3, |G+H|+2] \lhd (M + |G|/2)$.*

Lemma 2.2. *Let $m_1, \ldots, m_\alpha$ be positive integers. There is a labeling f of $G = \sum\limits_{i=1}^{\alpha} C_{4m_i}$ such that $I_f(G) = [3, |G|+2] \lhd (|G|/2 + 1)$.*

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Proof. There are labelings f_i of C_{4m_i} such that $I_{f_i}(C_{4m_i}) = [3, 4m_i+2]$, $1 \le i \le \alpha$. Applying Lemma 2.1 repeatedly, we get the lemma. □

3 Disjoint unions of cycles of orders congruent to 2 modulo 4

Let f be a labeling of a path $P = x_1x_2\cdots x_nx_{n+1}$, where $n \ge 1$. For convenience, we let $f(x_ix_{i+1}) = e_i$ $(1 \le i \le n)$, and $f^+(x_j) = a_j$ $(1 \le j \le n)$. Note that, in this case $f(x_1x_2) = a_1$ and $f^+(x_{n+1}) = e_n$. Suppose we know all the values of a_j. Clearly, e_i are determined uniquely. Let us present this result precisely:

Since we have the relations $a_j = e_j + e_{j-1}$, $2 \le j \le n$, $\sum\limits_{j=2}^{i}(-1)^j a_j = a_1 + (-1)^i e_i$, where $2 \le i \le n$. Thus

$$e_i = \sum_{j=1}^{i}(-1)^{j+i}a_j. \tag{3.1}$$

Note that

$$\begin{aligned} e_{2k} &= \sum_{j=1}^{k}(a_{2j} - a_{2j-1}) \\ e_{2k+1} &= a_1 - \sum_{j=2}^{2k+1}(-1)^j a_j = a_1 + \sum_{j=1}^{k}(a_{2j+1} - a_{2j}). \end{aligned} \tag{3.2}$$

Lemma 3.1. *Suppose $n \ge m$. There are labelings η_2 and η_3 of $G = C_{4m+2} + C_{4n+2}$ such that $I_{\eta_2}(G) = [2, |G| + 1]$ and $I_{\eta_3}(G) = [3, |G| + 2]$ with maximum edge label $|G|/2 + 1$.*

Proof. Let $C_{4m+2} = u_1u_2\cdots u_{4m+2}u_1$ and $C_{4n+2} = v_1v_2\cdots v_{4n+2}v_1$. We shall label C_{4m+2} first.

We divide the cycle C_{4m+2} into two (edge-disjoint) paths $P = u_1u_2\cdots u_{2m+1}u_{2m+2}$ and $P' = u_1u_{4m+2}u_{4m+1}\cdots u_{2m+3}u_{2m+2}$ of length $2m + 1$.

Firstly, we want to find a labeling ϕ for P satisfying that $\phi^+(u_1) = 1$ and $\phi^+(u_i) = 4(i-1)$ for $2 \le i \le 2m + 1$; and a labeling ψ for P' satisfying that $\psi^+(u_1) = 1$, $\psi^+(u_{4m+2}) = 3$ and $\psi^+(u_{4m+4-i}) = 4i - 3$ for $3 \le i \le 2m + 1$. By the discussion above, we know that ϕ and ψ exist. From (3.2) we have

$$\begin{aligned} \phi(u_{2m+1}u_{2m+2}) &= \phi^+(u_1) + \sum_{j=1}^{m} 4 = 1 + 4m, \\ \psi(u_{2m+3}u_{2m+2}) &= \psi^+(u_1) + (\psi^+(u_{4m+1}) - \psi^+(u_{4m+2})) + \sum_{j=2}^{m} 4 \\ &= 1 + 6 + 4(m-1) = 3 + 4m. \end{aligned}$$

Let $f = \phi \cup \psi$. Thus, $f^+(u_1) = 2$ and $f^+(u_{2m+2}) = 8m + 4$. Now $I_f(C_{4m+2}) = \{2, 3\} \cup (4[1, 2m+1]) \cup (4[2, 2m] + 1)$. One may check by (3.2) that all edge labels are positive and the maximum label is $4m + 3$ under f.

Now, we keep the labeling above for C_{4m+2} and try to label the cycle C_{4n+2}.

Case 1: $m = n$. Similarly, we divide the cycle C_{4n+2} into two paths $Q = v_1v_2 \cdots v_{2n+1}v_{2n+2}$ and $Q' = v_1v_{4n+2}v_{4n+1} \cdots v_{2n+3}v_{2n+2}$ of length $2n+1$. And then, we want to find a labeling ϕ for Q satisfying that $\phi^+(v_1) = 2$ and $\phi^+(v_i) = 4i - 2$ for $2 \le i \le 2n+1$; and a labeling ψ for Q' satisfying that $\psi^+(v_1) = 3$ and $\psi^+(u_{4n+4-i}) = 4i - 1$ for $2 \le i \le 2n+1$. Now

$$\phi(v_{2n+1}v_{2n+2}) = \phi^+(v_1) + \sum_{j=1}^{n} 4 = 2 + 4n,$$
$$\psi(v_{2n+3}v_{2n+2}) = \psi^+(v_1) + \sum_{j=1}^{n} 4 = 3 + 4n.$$

Let $g = \phi \cup \psi$. We have $g^+(v_1) = 5$, $g^+(v_{2n+2}) = 8n + 5$, all edge labels are positive and the maximum label is $4n+3$ under g. Moreover, $I_g(C_{4n+2}) = \{5\} \cup (4[1, 2n] + 2) \cup (4[1, 2n] + 3)$. Let $\eta_2 = f \cup g$. Here $I_{\eta_2}(C_{4m+2} + C_{4n+2}) = [2, 4m + 4n + 5] \triangleleft (4m + 3)$.

Case 2: $m < n$. Let $n = m + k$, where $k \ge 1$. For convenience, we rewrite $C_{4n+2} = v_1 \cdots v_{2m+1}y_1 \cdots y_{2k}v_{2n+2}z_1 \cdots z_{2k}v_{4n-2m+3}v_{4n-2m+4} \cdots v_{4n+1}v_{4n+2}v_1$. Now we divide C_{4n+1} into $R = v_1 \cdots v_{2m+1}y_1 \cdots y_{2k}v_{2n+2}$ and $R' = v_1v_{4n+2}v_{4n+1} \cdots v_{4n-2m+3}z_{2k} \cdots z_1v_{2n+2}$.

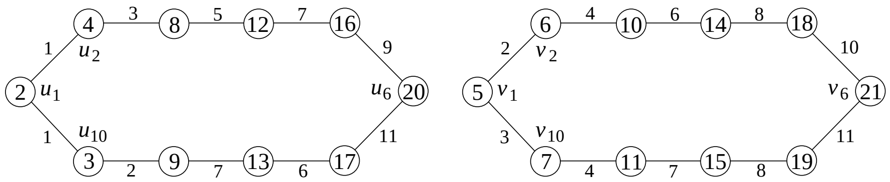
We find a labeling ϕ for R satisfying that $\phi^+(v_1) = 2$, $\phi^+(v_i) = 4i - 2$ for $2 \le i \le 2m+1$ and $\phi^+(y_i) = 8m + 3 + 2i$ for $1 \le i \le 2k$; and a labeling ψ for R' satisfying that $\psi^+(v_1) = 3$, $\psi^+(u_{4n+4-i}) = 4i - 1$ for $2 \le i \le 2m+1$ and $\psi^+(z_{2k-i+1}) = 8m + 4 + 2i$ for $1 \le i \le 2k$. Now

$$\phi(y_{2k}v_{2n+2}) = \phi^+(v_1) + \sum_{j=1}^{m} 4 + \sum_{j=1}^{k} 2 = 2 + 4m + 2k,$$
$$\psi(z_1v_{2n+2}) = \psi^+(v_1) + \sum_{j=1}^{m} 4 + \sum_{j=1}^{k} 2 = 3 + 4m + 2k.$$

Let $g = \phi \cup \psi$. Since the required vertex label set for $\{v_i \mid 2 \le i \le 2m+1, 4n-2m+3 \le i \le 4n+2\}$ is the same as Case 1 and $g^+(v_1) = 5$, $[2, 8m+4] \subset I_{f\cup g}(C_{4m+2} + C_{4n+2})$. From the requirement of vertices y_j's and z_j's, we get that $[8m+5, 4m+4n+4] \subset I_{f\cup g}(C_{4m+2} + C_{4n+2})$. Finally we have $g^+(v_{2n+1}) = 8m + 4k + 5 = 4m + 4n + 5$. Let $\eta_2 = f \cup g$. Here $I_{\eta_2}(C_{4m+2} + C_{4n+2}) = [2, 4m + 4n + 5]$. Moreover, one may check that all edge labels are positive and the maximum label is $4m + 2k + 3 = 2m + 2n + 3$ under η_2.

We define a new labeling η_3 from η_2 by increasing the label of each edge u_iu_{i+1} and v_jv_{j+1} for odd i $(1 \le i \le 4m+1)$ and odd j $(1 \le j \le 4n+1)$ by 1. Hence, $I_{\eta_3}(G) = [3, |G| + 2] \triangleleft (2m + 2n + 3)$. $\square$

Example 3.1. Here is a labeling η_2 for $C_{10} + C_{10} = 2C_{10}$ such that $I_{\eta_2}(2C_{10}) = [2, 21]$.



For $C_{10} + C_{18}$, we keep the labeling for the first C_{10} in $2C_{10}$ and the labeling for C_{18} is:

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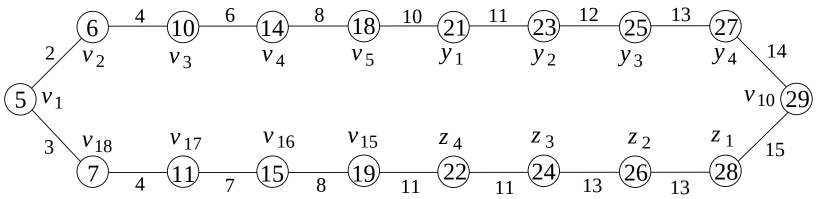
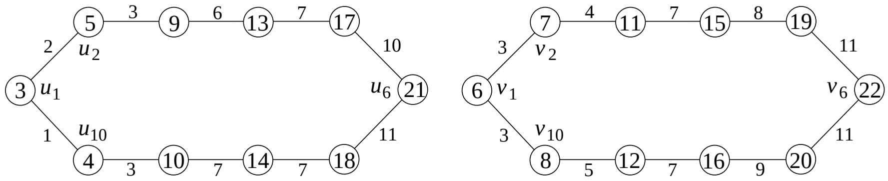
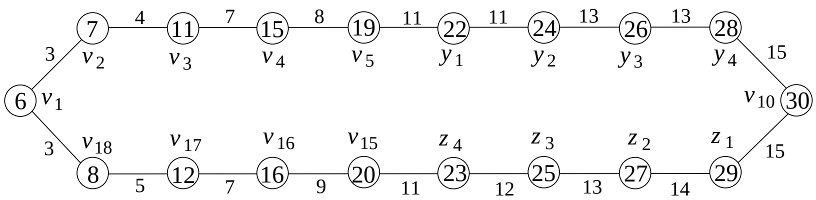


Now, one can see that $I_{\eta_2}(C_{10}+C_{18}) = [2, 29]$.

Example 3.2. Here is a labeling η_3 for $C_{10} + C_{10} = 2C_{10}$ such that $I_{\eta_3}(2C_{10}) = [3, 22]$.



For $C_{10} + C_{18}$, we keep the labeling for the first C_{10} in $2C_{10}$ and the labeling for C_{18} is:



Now, one can see that $I_{\eta_3}(C_{10}+C_{18}) = [3, 30]$.

4 Disjoint unions of cycles of orders congruent to 1 modulo 4

Lemma 4.1. *There is a labeling f_m for C_{4m+1} such that $I_f(C_{4m+1}) = [3, 4m+4] \setminus \{4m+1\} \triangleleft (2m+2)$.*

Proof. Let $C_{4m+1} = u_1 u_2 \cdots u_{4m+1} u_{4m+2}$, where $u_{4m+2} = u_1$. Define

$$f_m(u_i u_{i+1}) = \begin{cases} 1 + 2\lfloor \frac{i}{2} \rfloor, & \text{if } 1 \le i \le 2m; \\ 2m+2, & \text{if } i = 2m+1, 2n+2; \\ 4m+3-i, & \text{if } 2m+3 \le i \le 4n+1. \end{cases}$$

We obtain that

$$f_m^+(u_i) = \begin{cases} 3, & i = 1; \\ 2i, & \text{if } 2 \le i \le 2m; \\ 4m+3, & \text{if } i = 2m+1; \\ 4m+4, & \text{if } i = 2m+2; \\ 4m+2, & i = 2n+3; \\ 8m+7-2i, & \text{if } 2m+4 \le i \le 4m+1. \end{cases}$$

Hence $I_{f_m}(C_{4m+1}) = [3, 4m+4] \setminus \{4m+1\} \triangleleft (2m+2)$. $\square$

Lemma 4.2. *There is a labeling f_n for C_{4n+1} such that $I_{f_n}(C_{4n+1}) = \{3\} \cup [7, 4n+7] \setminus \{4n+4\} \lhd (2n+5)$.*

Proof. Let $C_{4n+1} = w_1 w_2 \cdots w_{4n+1} w_{4n+2}$, where $w_{4n+2} = w_1$. Define

$$f_n(w_i w_{i+1}) = \begin{cases} i, & \text{if } i \text{ is odd and } 1 \le i \le 2n-1; \\ 4+i, & \text{if } i \text{ is even and } 2 \le i \le 2n; \\ 2n+2, & \text{if } i = 2n+1; \\ 2n+5, & \text{if } i = 2n+2; \\ 4n+3-i, & \text{if } i \text{ is odd and } 2n+3 \le i \le 4n+1; \\ 4n+6-i, & \text{if } i \text{ is even and } 2n+4 \le i \le 4n. \end{cases}$$

We obtain that

$$f_n^+(v_i) = \begin{cases} 3, & i = 1; \\ 3+2i, & \text{if } 2 \le i \le 2n; \\ 4n+6, & \text{if } i = 2n+1; \\ 4n+7, & \text{if } i = 2n+2; \\ 4n+5, & \text{if } i = 2n+3; \\ 8n+10-2i, & \text{if } 2n+4 \le i \le 4n+1. \end{cases}$$

Hence $I_{f_n}(C_{4n+1}) = \{3\} \cup [7, 4n+7] \setminus \{4n+4\} \lhd (2n+5)$. $\square$

Lemma 4.3. *There is a labeling f_s for C_{4s+1} such that $I_{f_s}(C_{4s+1}) = \{2\} \cup [6, 4s+5] \lhd (2s+4)$.*

Proof. Let $C_{4s+1} = v_1 v_2 \cdots v_{4s+1} v_{4s+2}$, where $v_{4s+2} = v_1$. Define

$$f_s(v_i v_{i+1}) = \begin{cases} i, & \text{if } i \text{ is odd and } 1 \le i \le 2s+1; \\ 3+i, & \text{if } i \text{ is even and } 2 \le i \le 2s; \\ 4s+2-i, & \text{if } i \text{ is odd and } 2s+3 \le i \le 4s+1; \\ 4s+6-i, & \text{if } i \text{ is even and } 2s+2 \le i \le 4s. \end{cases}$$

We obtain that

$$f_s^+(v_i) = \begin{cases} 2, & i = 1; \\ 2+2i, & \text{if } 2 \le i \le 2s+1; \\ 4s+5, & \text{if } i = 2s+2; \\ 8s+9-2i, & \text{if } 2s+3 \le i \le 4s+1. \end{cases}$$

Hence $I_{f_s}(C_{4s+1}) = \{2\} \cup [6, 4s+5] \lhd (2s+4)$. $\square$

Lemma 4.4. *Let m, n, s, t be positive integers. There is a labeling η of $G = C_{4m+1} + C_{4n+1} + C_{4s+1} + C_{4t+1}$ such that $I_\eta(G) = [3, |G|+2] \lhd (|G|/2+1)$.*

Proof. From Lemmas 4.1, 4.2, 4.3 and 1.4, we have a labelings f_m, f_n, f_s and f_t such that $I_{f_m}(C_{4m+1}) = [3, 4m+4] \setminus \{4m+1\}$, $I_{f_n}(C_{4n+1}) = \{3\} \cup [7, 4n+7] \setminus \{4n+4\}$, $I_{f_s}(C_{4s+1}) = \{2\} \cup [6, 4s+5]$ and $I_{f_t}(C_{4t+1}) = [2, 4t+2]$.

Let $g_n = f_n + (2m-1)$, $g_s = f_s + (2m+2n)$ and $g_t = f_t + (2m+2n+2s+2)$. Combining f_m, g_n, g_s and g_t as a labeling η for $C_{4m+1} + C_{4n+1} + C_{4s+1} + C_{4t+1}$ we have the required labeling. Clearly, the maximum edge label is $2(m+n+s+t)+3$. $\square$

5 Disjoint unions of cycles of orders congruent to 3 modulo 4

It is known that C_3 is not $\mathbb{Z}_3$-antimagic [2, Theorem 3]. Following we will define some labelings for disjoint union of cycles whose orders are congruence 3 modulo 4. In particular, there are some labelings of disjoint unions of 3-cycles that are useful, even though they are not antimagic labelings.

Lemma 5.1. *There are labelings g and h of C_{4n-1} such that $I_g(C_{4n-1}) = \{3\} \cup [7, 4n+4] \lhd (2n+4)$ and $I_h(C_{4n-1}) = [2, 4n+1] \setminus \{4n-2\} \lhd (2n+1)$.*

Proof. Let $C_{4n-1} = u_1 u_2 \cdots u_{4n-1} u_{4n}$, where $u_{4n} = u_1$. Define

$$g(u_i u_{i+1}) = \begin{cases} i & \text{if } i \text{ is odd and } 1 \le i \le 2n-1; \\ 4+i & \text{if } i \text{ is even and } 2 \le i \le 2n; \\ 4n+1-i & \text{if } i \text{ is odd and } 2n+1 \le i \le 4n-1; \\ 4n+4-i & \text{if } i \text{ is even and } 2n+2 \le i \le 4n-2. \end{cases}$$

We obtain that

$$g^+(u_i) = \begin{cases} 3, & i = 1; \\ 3+2i, & \text{if } 2 \le i \le 2n; \\ 4n+4, & \text{if } i = 2n+1; \\ 8n+6-2i, & \text{if } 2n+2 \le i \le 4n-1. \end{cases}$$

Hence $I_g(C_{4n-1}) = \{3\} \cup [7, 4n+4]$.

When $n \ge 2$. Define

$$h(u_i u_{i+1}) = \begin{cases} i & \text{if } i \text{ is odd and } 1 \le i \le 2n-3; \\ 1+i & \text{if } i \text{ is even and } 2 \le i \le 2n-2; \\ 1+i & \text{if } 2n-1 \le i \le 2n; \\ 4n-i & \text{if } 2n+1 \le i \le 4n-1. \end{cases}$$

We obtain that

$$h^+(u_i) = \begin{cases} 2, & i = 1; \\ 2i, & \text{if } 2 \le i \le 2n-2; \\ 4n-1, & \text{if } i = 2n-1; \\ 4n+1, & \text{if } i = 2n; \\ 4n, & \text{if } i = 2n+1; \\ 8n+1-2i, & \text{if } 2n+2 \le i \le 4n-1. \end{cases}$$

Hence $I_h(C_{4n-1}) = [2, 4n+1] \setminus \{4n-2\}$. Note that, when $n = 1$, $h = f$, where f is defined in Lemma 1.4. Hence $I_h(C_3) = [3, 5]$. So it still holds. $\square$

Corollary 5.2. *Let $G = C_{4m-1} + C_{4n-1}$, where $m, n \ge 1$. There is a labeling f_3 such that $I_{f_3}(G) = [3, |G|+3] \setminus \{|G|\} \lhd (|G|/2+2)$.*

Proof. Let f be the labeling defined in Lemma 1.4 and h be the labeling defined in Lemma 5.1. Let $f_3 = f \cup (h+2m)$. Here $I_{f_3}(G) = [3, 4m+4n+1] \setminus \{4m+4n-2\}$ with maximum edge label $2m+2n+1$. $\square$

Lemma 5.3. *There is a labeling ℓ of C_{4n-1} such that $I_\ell(C_{4n-1}) = \{2\} \cup [6, 4n+4] \setminus \{4n+1\} \lhd (2n+4)$.*

Proof. Keeping the notation as Lemma 5.1, we define

$$\ell(u_iu_{i+1}) = \begin{cases} i & \text{if } i \text{ is odd and } 1 \le i \le 2n-1; \\ 3+i & \text{if } i \text{ is even and } 2 \le i \le 2n-2; \\ 2n & \text{if } i = 2n+1; \\ 4n-i & \text{if } i \text{ is odd and } 2n+3 \le i \le 4n-1; \\ 4n+4-i & \text{if } i \text{ is even and } 2n \le i \le 4n-2. \end{cases}$$

We obtain that

$$\ell^+(u_i) = \begin{cases} 2 & \text{if } i = 1; \\ 2+2i, & \text{if } 2 \le i \le 2n-1; \\ 4n+3, & \text{if } i = 2n; \\ 4n+4, & \text{if } i = 2n+1; \\ 4n+2, & \text{if } i = 2n+2; \\ 8n+5-2i, & \text{if } 2n+3 \le i \le 4n-1. \end{cases}$$

Hence $I_\ell(C_{4n-1}) = \{2\} \cup [6, 4n+4] \setminus \{4n+1\}$. $\square$

Corollary 5.4. *Let $G = C_{4m-1} + C_{4n-1}$, where $m \le n$ and $n \ge 2$. There is a labeling f_6 such that $I_{f_6}(G) = \{2\} \cup [6, |G|+4] \lhd (|G|/2+4)$.*

Proof. Let g and ℓ be labelings defined in Lemmas 5.1 and 5.3, respectively. Hence $I_g(C_{4m-1}) = \{3\} \cup [7, 4m+4]$ and $I_\ell(C_{4n-1}) = \{2\} \cup [6, 4n+4] \setminus \{4n+1\}$. Let $f_6 = \ell \cup (g + (2n-1))$. Here $I_{f_6}(G) = \{2\} \cup [6, 4m+4n+2] \lhd (2m+2n+3)$. $\square$

Lemma 5.5. *There is a labeling f_8 of $2C_3$ such that $I_{f_8}(2C_3) = \{4\} \cup [8, 12] \lhd (8)$.*

Proof. Let f_8 be the labeling of $2C_3$ by labeling the edges of the first C_3 by 1,3,8 and the edges of the second C_3 by 3,5,7. Then $I_{f_8}(2C_3) = \{4\} \cup [8, 12]$. $\square$

Note that f_8 is neither a $\mathbb{Z}_7$-antimagic labeling nor a $\mathbb{Z}_8$-antimagic labeling, since 0 would be an edge label in each.

Corollary 5.6. *Let $G = C_{4m-1} + C_{4n-1} + C_{4s-1}$, where $m \le n \le s$. There is a labeling η_1 such that $I_{\eta_1}(G) = [2, |G|+1]$ when $s \ge 2$ and $I_{\eta_1}(3C_3) = [4, 12]$. The maximum edge label is $(|G|+1)/2+2$.*

Proof. Let h be the labeling of C_{4m-1} defined in Lemma 5.1 and f_6 be the labeling of $C_{4n-1} + C_{4s-1}$ defined in Lemma 5.5.

When $s \ge 2$. Let $\eta_1 = h \cup (f_6 + (2m-2))$. Hence $I_{\eta_1}(G) = [2, 4m+4n+4s-2] \lhd (2m+2n+2s+1)$.

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When $s = 1$. In this case $G = 3C_3$. We label the edges of these three C_3 by $1, 4, 3$; $2, 6, 4$; $4, 7, 5$ accordingly. Denote this labeling by η_1. Then $I_{\eta_1}(3C_3) = [4, 12]$. Note that the maximum edge label is still $2m + 2n + 2s + 1$. □

Lemma 5.7. *Let $G = C_{4m-1} + C_{4n-1} + C_{4s-1} + C_{4t-1}$, where $m \le n \le s \le t$. There is a labeling ϕ such that $I_\phi(G) = [3, |G| + 2] \triangleleft (|G|/2 + 3)$ when $t \ge 2$ and $I_\phi(4C_3) = [3, 14] \triangleleft (|4C_3|/2 + 2)$.*

Proof. Consider labelings f for C_{4m-1} defined in Lemma 1.4 and η_1 for $C_{4n-1}+C_{4s-1}+C_{4t-1}$ defined in Corollary 5.6.

When $t \ge 2$. Let $\phi = f \cup (\eta_1 + 2m)$. Here $I_\phi(G) = [3, |G| + 2] \triangleleft (|G|/2 + 3)$. When $t = 1$. Let $\phi = f \cup (\eta_1 + 1)$. Here $I_\phi(G) = [3, 14] \triangleleft (8)$. □

6 Main result

Before considering the general case, we introduce another lemma first.

Lemma 6.1. *There is a labeling σ_3 for $G = C_{4m+1} + C_{4n-1}$ such that $I_{\sigma_3}(G) = [3, |G| + 2] \triangleleft (|G|/2 + 1)$.*

Proof. From Lemma 1.4 there are labelings f and g such that $I_f(C_{4m+1}) = [2, 4m+2] \triangleleft (2m+1)$ and $I_g(C_{4n-1}) = [3, 4n+1] \triangleleft (2n+1)$. Let $\sigma_3 = g \cup (f + 2n)$. Here $I_{\sigma_3} = [3, |G|+2] \triangleleft (|G|/2+1)$. □

Now consider a disjoint union of cycles

$$G = \sum_{i=1}^{\alpha} C_{4m_i} + \sum_{j=1}^{\beta} C_{4n_j+1} + \sum_{k=1}^{\gamma} C_{4s_k+2} + \sum_{l=1}^{\delta} C_{4t_l-1},$$

where $\alpha, \beta, \gamma, \delta \ge 0$ and m_i, n_j, s_k, t_l are positive. Note that, as usual, a summation is empty if its upper limit is less than its lower limit. Since the integer-antimagic spectra of single cycle has been determined [2], we may assume that G contains at least two cycles.

Suppose $\beta = 4\beta' + \beta_0$, $\gamma = 2\gamma' + \gamma_0$, and $\delta = 4\delta' + \delta_0$, where $0 \le \beta_0, \delta_0 \le 3$ and $0 \le \gamma_0 \le 1$. When $\beta_0 \ge \delta_0 \ge 1$, let $G_0 = \sum\limits_{l=4\delta'+1}^{\delta} (C_{4n_l+1} + C_{4t_l-1})$. Similarly, when $\delta_0 \ge \beta_0 \ge 1$, let $G_0 = \sum\limits_{j=4\beta'+1}^{\beta} (C_{4n_j+1} + C_{4t_j-1})$.

Let $G_1 = \sum\limits_{i=1}^{\alpha} C_{4m_i} + \sum\limits_{j=1}^{4\beta'} C_{4n_j+1} + \sum\limits_{k=1}^{2\gamma'} C_{4s_k+2} + \sum\limits_{l=1}^{4\delta'} C_{4t_l-1} + G_0$. We may rewrite the remaining part of G as $G - G_1 = \sum\limits_{j=1}^{\beta_1} C_{4p_j+1} + \sum\limits_{k=1}^{\gamma_0} C_{4r_k+2} + \sum\limits_{l=1}^{\delta_1} C_{4q_l-1}$, where $0 \le \gamma_0 \le 1$, $0 \le \beta_1, \delta_1 \le 3$ and at least one of β_1 and δ_1 is zero.

From Lemmas 2.2, 4.4, 3.1, 5.7 and 6.1 and applying Lemma 2.1 repeatedly, there are labelings for $\sum\limits_{i=1}^{\alpha} C_{4m_i}$, $\sum\limits_{j=1}^{4\beta'} C_{4n_j+1}$, $\sum\limits_{k=1}^{2\gamma'} C_{4s_k+1}$, $\sum\limits_{l=1}^{4\delta'} C_{4t_l-1}$ and G_0 such that the range of each labeling is $[3, |H| + 2] \triangleleft (|H|/2 + 1)$, where H is one of the considered graphs.

Suppose G_1 exists. Applying Lemma 2.1 on G_1 repeatedly, we have a labeling θ of G_1 such that $I_\theta(G_1) = [3, |G_1| + 2] \triangleleft (|G_1|/2 + 1)$.

Following we will deal with the remaining part $G - G_1$ if any. So we assume that at least one of $\beta_1, \gamma_0, \delta_1$ is nonzero. Let

$$H_1 = \sum_{j=1}^{\beta_1} C_{4p_j+1} + \sum_{l=1}^{\delta_1} C_{4q_l-1}, \text{ if } \gamma_0 = 0,$$
$$H_2 = H_1 + C_{4r+2} \text{ for some } r \geq 1, \text{ if } \gamma_0 = 1,$$

where $0 \leq \beta_1, \delta_1 \leq 3$ and at least one of β_1 and δ_1 is zero.

Case 1: Consider the case when $\gamma_0 = 0$, that is $G = G_1 + H_1$.

1-1: Suppose $\beta_1 = 0$.

When $\delta_1 = 1$. By using the labeling f in Lemma 1.4 we have $I_f(H_1) = [3, |H_1| + 2]$. Let $\sigma = \theta \cup (f + |G_1|/2)$. Here $I_\sigma(G) = [3, |G| + 2] \triangleleft (|G| + 1)/2 + 1)$.

When $\delta_1 = 2$. By using the labeling f_3 in Corollary 5.2, we have $I_{f_3}(H_1) = [3, |H_1| + 3] \setminus \{|H_1|\} \triangleleft (|H_1|/2 + 2)$. Let $\sigma = \theta \cup (f_3 + |G_1|/2)$. Hence $I_\sigma(G) = [3, |G| + 3] \setminus \{|G| + 2\} \triangleleft (|G|/2 + 2)$.

When $\delta_1 = 3$. Choose the labeling η_1 in Corollary 5.6. Let $\sigma = \eta_1 \cup (\theta + x)$, where x is $(|H_1| - 1)/2$ or $(|H_1| + 1)/2$ for $H_1 \neq 3C_3$ or $H_1 = 3C_3$, respectively. Here $I_\sigma(G) = [2, |G| + 1]$ or $[4, |G| + 3]$ and the maximum label at most $(|G| + 1)/2 + 1$.

1-2: Suppose $\delta_1 = 0$.

When $\beta_1 = 1$. Let f be the labeling in Lemma 1.4. Let $\sigma = f \cup (\theta + (|H_1| - 1)/2)$. Hence, $I_\sigma(G) = [2, |G| + 1] \triangleleft ((|G| - 1)/2)$.

When $\beta_1 = 2$. Combining the labeling in Lemma 4.1 with a translation of the labeling in Lemma 4.2 we have a labeling f' for H_1 such that $I_{f'}(H_1) = [3, |H_1| + 3] \setminus \{|H_1|\} \triangleleft (|H_1|/2 + 3)$. Let $\sigma = \theta \cup (f' + |G_1|/2)$. Hence, $I_\sigma(G) = [3, |G| + 3] \setminus \{|G|\} \triangleleft (|G|/2 + 3)$.

When $\beta_1 = 3$. Using the labelings defined in Lemma 4.1, Lemma 4.2 and Lemma 4.3, we have a labeling f' for H_1 such that $I_{f'}(H_1) = [3, |H_1| + 2] \triangleleft ((|H_1| + 1)/2 + 1)$. Let $\sigma = \theta \cup (f' + |G_1|/2)$. Hence, $I_\sigma(G) = [3, |G| + 2] \triangleleft ((|G| + 1)/2 + 1)$.

Case 2: Consider the case when $\gamma_0 = 1$, that is $G = G_1 + H_1 + C_{4r+2}$. Here H_1 may not exist.

2-1: When $(\beta_1, \delta_1) = (0, 0)$. In this case, $G = G_1 + C_{4r+2}$. We have the labeling θ for G_1 such that $I_\theta(G_1) = [3, |G_1| + 2] \triangleleft (|G_1|/2 + 1)$. From Lemma 1.4 there is a labeling f for C_{4r+2} such that $I_f(C_{4r+2}) = [3, 4r + 5] \setminus \{4r + 2\} \triangleleft (2r + 3)$. Let $\rho = \theta \cup (f + |G_1|/2)$. Then $I_\rho(G) = [3, |G| + 3] \setminus \{|G|\} \triangleleft (|G|/2 + 2)$.

2-2: When $(\beta_1, \delta_1) = (0, 2)$. In this case, $G = G_1 + C_{4r+2} + C_{4q_1-1} + C_{4q_2-1}$. Let ρ be the labeling in Case 2-1. Let $L_1 = C_{4q_1-1} + C_{4q_2-1}$.

Suppose $L_1 \neq 2C_3$. From Corollary 5.4 we have a labeling f_6 for $L_1 =$ such that $I_{f_6}(L_1) = \{2\} \cup [6, |L_1| + 4] \triangleleft (|L_1|/2 + 4)$. Let $\rho_1 = \rho \cup (f_6 + |G_1 + C_{4r+2}|/2 - 1)$. Here $I_{\rho_1}(G) = [3, |G| + 2] \triangleleft (|G|/2 + 3)$.

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Suppose $L_1 = 2C_3$. From Lemma 5.5 we have a labeling f_8 for L_1 such that $I_{f_8}(L_1) = \{4\} \cup [8, 12] \triangleleft (8)$. Let $\rho_1 = \rho \cup (f_8 + |G_1 + C_{4r+2}|/2 - 2)$. Here $I_{\rho_1}(G) = [3, |G| + 2] \triangleleft (|G|/2 + 3)$.

2-3: When $(\beta_1, \delta_1) = (2, 0)$. In this case, $G = G_1 + C_{4r+2} + C_{4p_1+1} + C_{4p_2+1}$. Let $L_2 = C_{4p_1+1} + C_{4p_2+1}$. Combining the labeling in Lemma 4.3 with a translation of the labeling in Lemma 1.4, we have a labeling f for L_2 such that $I_f(L_2) = \{2\} \cup [6, |L_2| + 4] \triangleleft (|L_2|/2 + 4)$. Similar to Case 2-2 we have a labeling ρ_2 for G such that $I_{\rho_2}(G) = [3, |G| + 2] \triangleleft (|G|/2 + 3)$.

2-4: When $(\beta_1, \delta_1) = (0, 1)$ or $(3, 0)$. Let σ be the labeling of $G_1 + H_1$ in Case 1 and h be the labeling in Lemma 1.7. Let $\rho_3 = \sigma \cup (h + (|G_1 + H_1| + 1)/2)$. Then $I_{\rho_3}(G) = [3, |G| + 3] \setminus \{|G|\} \triangleleft ((|G| + 1)/2 + 2)$.

2-5: When $(\beta_1, \delta_1) = (0, 3)$ or $(1, 0)$. Let σ be the labeling of $G_1 + H_1$ in Case 1 and f be the labeling in Lemma 1.4.

Suppose $H_1 \neq 3C_3$. Let $\rho_4 = \sigma \cup (f + (|G_1 + H_1| - 1)/2)$. Then $I_\rho(G) = [2, |G| + 2] \setminus \{|G| - 1\} \triangleleft ((|G| + 1)/2 + 1)$.

Suppose $H_1 = 3C_3$. Let $\rho_4 = \sigma \cup (f + (|G_1 + H_1| + 1)/2)$. Then $I_\rho(G) = [4, |G| + 4] \setminus \{|G| + 1\} \triangleleft ((|G| + 1)/2 + 2)$.

Suppose G_1 does not exist. Then the proof is similar and simpler than the case when G_1 exists. So we omit here.

From the discussion in this paper, we can see that all the edge labels are positive and at most $\lfloor |G|/2 \rfloor + 3$, where G is the considered graph. If $|G| \geq 7$, then $\lfloor |G|/2 \rfloor + 3 < |G|$. So, for these cases, all labelings are proper when we take modulo k, where $k \geq |G|$. If $3 \leq |G| \leq 6$ and G contains at least two cycles, then G must be $2C_3$. Corollary 5.2 shows that $I_{f_3}(2C_3) = [3, 9] \setminus \{6\} \triangleleft (5)$. Thus, the labeling is proper.

Applying Proposition 1.3 we get

Theorem 6.2. *Suppose G is a disjoint union of cycles of order p. Then*

$$IAM(G) = \begin{cases} [4, \infty) & \text{if } p = 3, \\ [p, \infty) & \text{if } p \not\equiv 2 \pmod 4 \text{ and } p \neq 3, \\ [p + 1, \infty) & \text{if } p \equiv 2 \pmod 4. \end{cases}$$

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